A family of generalized gamma convoluted variables

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Abstract This paper consists of three parts : in the first part, we describe a family of generalized gamma convoluted (abbreviated as GGC) variables. In the second part, we use this description to prove that several r.v.'s, related to the length of excursions away from 0 for a recurrent linear diffusion on $\mathbb{R}_+$, are GGC. Finally, in the third part, we apply our results to the case of Bessel processes with dimension $d = 2(1-\alpha)$ $(0 < d < 2$, or $0 < \alpha < 1)$.

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0 Notation and Introduction

0.1 Let $l : \mathbb{R}_+ \to \mathbb{R}_+$ denote a Borel function such that :

$$\int_0^\infty \frac{l(z)}{z}\, dz < \infty \tag{0.1}$$

Without loss of generality, we assume that :

$$\int_0^\infty \frac{l(z)}{z}\, dz = 1 \tag{0.2}$$

With l, we associate a r.v. Y on $\mathbb{R}_+$ whose probability density f_Y is given by :

$$f_Y(u) = \int_0^\infty e^{-uz}\, l(z) dz \qquad (u \geq 0) \tag{0.3}$$

Indeed, due to (0.2), we get :

$$\int_0^\infty f_Y(u) du = \int_0^\infty du \int_0^\infty e^{-uz}\, l(z) dz = \int_0^\infty \frac{l(z)}{z}\, dz = 1 \tag{0.4}$$

To emphasize the relation between Y and l, we shall (sometimes) write Y_l.

We denote by $\varphi_l \equiv \varphi_{Y_l}$ the Laplace transform of Y_l :

$$\begin{aligned}\varphi_l(\lambda) = \varphi_{Y_l}(\lambda) &= E(e^{-\lambda Y_l}) = \int_0^\infty e^{-\lambda u} f_{Y_l}(u) du \\ &= \int_0^\infty \frac{l(z)}{\lambda + z}\, dz \end{aligned}\tag{0.5}$$

Thus, since f_{Y_l} is the Laplace transform of l, φ_l is the Stieltjes transform of l.

0.2 A reminder about GGC variables

Let μ denote a positive, σ-finite measure on $\mathbb{R}_+$. We recall $\big($see [Bon]$\big)$ that a positive r.v. Y is a GGC variable with Thorin measure μ if :

$$E(e^{-\lambda Y}) = \exp\left\{-\int_0^\infty (1-e^{-\lambda x})\,\frac{dx}{x}\int_0^\infty e^{-xz}\mu(dz)\right\} \qquad (\lambda \geq 0) \tag{0.6}$$

Such a r.v. is self-decomposable, hence infinitely divisible.

The GGC r.v.'s Y whose Thorin measure μ has a finite total mass, equal to m, are characterized by $\big($see [JRY]$\big)$:

$$E(e^{-\lambda Y}) = \exp\left\{-m\int_0^\infty (1-e^{-\lambda x})\,\frac{dx}{x}\,E(e^{-xG})\right\} \tag{0.7}$$

where G is an $\mathbb{R}_+$-valued r.v. such that $E\big(\log^+(1/G)\big) < \infty$.

Such a r.v. is a gamma-m mixture, i.e. it satisfies [1] :

$$Y \overset{\text{(law)}}{=} \gamma_m \cdot Z \tag{0.8}$$

where γ_m is a gamma variable with parameter m, independent from the $\mathbb{R}_+$-valued variable Z. We note that any r.v. which is a gamma-m mixture is also a gamma-m' mixture, for any $m' > m$, since there is the identity :

$$\gamma_m \overset{\text{(law)}}{=} \gamma_{m'} \cdot \beta_{m,\,m'-m} \tag{0.9}$$

where $\gamma_{m'}$ is a gamma variable with parameter m' and $\beta_{m,\,m'-m}$ is a beta variable with parameters $(m,\, m'-m)$ independent from $\gamma_{m'}$.

We also recall $\big($see [Bon], p. 51$\big)$ that the parameter m of a GGC r.v. Y, with Thorin measure with total mass m, may be obtained from the formula :

$$m = \sup\left\{\delta \geq 0\ ;\ \lim_{u\downarrow 0_+} \frac{f_Y(u)}{u^{\delta-1}} = 0\right\} \tag{0.10}$$

1 A family of GGC variables

The aim of this part is to present a sufficient condition on l which implies that the associated variable Y_l is GGC.

Definition 1 A function l which satisfies (0.1) belongs to the class $\mathcal{C}$ if there exist $a \geq 0$, $b > a$, $\sigma \geq 0$ and $\theta : \mathbb{R}_+ \to \mathbb{R} \cup (+\infty)$ a Borel, decreasing function, which is identically equal to $+\infty$ on $[0, a[$, such that :

$$l(z) = \exp\left\{\sigma + \int_b^z \frac{\theta(y)}{y}\,dy\right\} \tag{1.1}$$

[1] It would be more correct to say that : the law of such a r.v. is a gamma-m mixture ; however, such abuse is usual, and should not lead to confusion.

Of course, if (1.1) is satisfied with $a > 0$, then the function l is identically 0 on $[0, a[$.
On the other hand, if l is identically 0 on $[0, a[$ and differentiable on $]a, \infty[$, then l belongs to the class $\mathcal{C}$ if and only if the function :

$$y \to y\ (\log l)'(y) := \theta(y) \tag{1.2}$$

is decreasing on $[a, \infty[$.

The following properties are elementary :

- If $l \in \mathcal{C}$, then for every $u > 0$, $x \to l(ux) \in \mathcal{C}$ (1.3)
- If $l_1, l_2 \in \mathcal{C}$, then $l_1 \cdot l_2 \in \mathcal{C}$ (1.4)
- For every α real, $x \to x^\alpha \in \mathcal{C}$ (1.5)
- For every $k < 0$ and $\gamma \geq 0$, $x \to (x + \gamma)^k \in \mathcal{C}$ (1.6)

Theorem 2 *Let l which satisfies (0.2) and belongs to $\mathcal{C}$, and let Y_l denote the r.v. associated with l. Then :*
Y_l is a GGC r.v. whose Thorin measure μ has total mass m smaller than or equal to 1. In other terms, there exists a r.v. G taking values in $\overline{\mathbb{R}}_+$, and satisfying $E\big(\log^+(1/G)\big) < \infty$ and $m \leq 1$ such that :

$$E(e^{-\lambda Y_l}) = \exp\left\{-m \int_0^\infty (1 - e^{-\lambda x})\, \frac{dx}{x}\, E(e^{-xG})\right\} \qquad (\lambda \geq 0) \tag{1.7}$$

Proof of Theorem 2
1. It suffices to show that Y_l is GGC since, if so, then the total mass m of its Thorin measure equals, from (0.3) and (0.10) :

$$m = \sup\left\{\delta \geq 0\ ;\ \lim_{u \downarrow 0_+} \frac{1}{u^{\delta - 1}} \int_0^\infty e^{-uz}\, l(z) dz = 0\right\}$$

and, of course, $m \leq 1$ since, for $\delta = 1$:

$$\frac{1}{u^{\delta-1}} \int_0^\infty e^{-uz}\, l(z) dz = \int_0^\infty e^{-uz}\, l(z) dz \underset{u \downarrow 0_+}{\longrightarrow} \int_0^\infty l(z) dz > 0$$

2. To show that Y_l is GGC, we shall use the following characterization (see [Bon], Th. 6.1.1, p. 90) of these r.v.'s :
Y is GGC if and only if its Laplace transform φ_Y is hyperbolically completely monotone, that is it satisfies : for every $u > 0$, the function H_u, defined by :

$$H_u(w) = \varphi_Y(uv) \cdot \varphi_Y\left(\frac{u}{v}\right), \quad \text{where } w = v + \frac{1}{v} \tag{1.8}$$

is a completely monotone function, i.e. it is the Laplace transform of a positive measure carried by $\mathbb{R}_+$.
In our framework, this criterion becomes : for every $u > 0$, H_u is completely monotone with, from (0.5) :

$$H_u(w) = \int_0^\infty \int_0^\infty \frac{l(x) l(y)}{(x + uv)(y + \frac{u}{v})}\, dx\, dy \qquad \left(w = v + \frac{1}{v}\right) \tag{1.9}$$

$$= \int_0^\infty \int_0^\infty \frac{l(ux) l(uy)}{(x + v)(y + \frac{1}{v})}\, dx\, dy \tag{1.10}$$

(after the change of variables $x = ux',\ y = uy'$).

Our aim being to show that the hypothesis : $l \in \mathcal{C}$ implies that H_u is completely monotone, and since $x \to l(ux)$ belongs to $\mathcal{C}$ if $l \in \mathcal{C}$ $\big($from (1.3)$\big)$, it suffices to see that the function H defined by :

$$H(w) := \int_0^\infty \int_0^\infty \frac{l(x)l(y)}{(x+v)(y+\frac{1}{v})}\, dx\, dy \qquad \left(w = v + \frac{1}{v}\right) \tag{1.11}$$

is completely monotone.

3. We show that H, defined by (1.11), is completely monotone :

i) We write :

$$\begin{aligned} H(w) &= \\ & \int_0^\infty \int_0^\infty \frac{l(x)l(y)}{(x+v)(y+\frac{1}{v})}\, dx\, dy = \frac{1}{2}\int_0^\infty \int_0^\infty l(x)l(y)\left[\frac{1}{(x+v)(y+\frac{1}{v})} + \frac{1}{(x+\frac{1}{v})(y+v)}\right] dx\, dy \\ & \qquad \text{(by symmetry)} \\ &= \frac{1}{2}\int_0^\infty \int_0^\infty l(x)l(y)\left[\frac{x^2-1}{xy-1}\cdot\frac{1}{x^2+xw+1} + \frac{y^2-1}{xy-1}\cdot\frac{1}{y^2+yw+1}\right] dx\, dy \qquad (1.12)\end{aligned}$$

(after reducing both reciprocals to the same denominator and decomposing into simple elements)

$$\begin{aligned} &= \frac{1}{2}\int_0^\infty \int_0^\infty l(x)l(y)dx\, dy \left[\frac{x^2-1}{xy-1}\int_0^\infty e^{-b(x^2+xw+1)}db + \frac{y^2-1}{xy-1}\int_0^\infty e^{-b(y^2+yw+1)}db\right] \\ &= \frac{1}{2}\int_0^\infty \int_0^\infty l(x)l(y)dx\, dy \cdot \left[\frac{x^2-1}{xy-1}\cdot\frac{1}{x}\int_0^\infty e^{-bw-b(x+\frac{1}{x})}db + \frac{y^2-1}{xy-1}\,\frac{1}{y}e^{-bw-b(y+\frac{1}{y})}db\right]\end{aligned}$$

(after making the change of variables $bx = b'$, $by = b'$)

$$= \int_0^\infty e^{-bw} db \left(\int_0^\infty \int_0^\infty l(x)l(y)\frac{x^2-1}{(xy-1)x}\, e^{-b(x+\frac{1}{x})}dx\, dy\right) \tag{1.13}$$

after interverting the orders of integration.

We note that the preceding computation is a little formal : we have transformed an absolutely convergent integral in an integral which is no longer absolutely convergent ; however, this does not matter for our purpose, as we shall soon gather the different terms in another way.

ii) Thus, we need to show, from (1.13), that, for every $b \geq 0$:

$$I_b := \int_0^\infty \int_0^\infty l(x)l(y)\,\frac{x^2-1}{(xy-1)x}\, e^{-b(x+\frac{1}{x})}dx\, dy \geq 0 \tag{1.14}$$

• Let us show (1.14). For this purpose, we define the 4 domains :

$$\mathcal{N}_1 = \left\{0 < x \leq 1,\ y > \frac{1}{x}\right\}, \qquad \mathcal{N}_2 = \left\{x \geq 1,\ y < \frac{1}{x}\right\}$$

$$\mathcal{P}_1 = \left\{x \geq 1,\ y > \frac{1}{x}\right\}, \qquad \mathcal{P}_2 = \left\{0 < x \leq 1,\ y < \frac{1}{x}\right\}$$

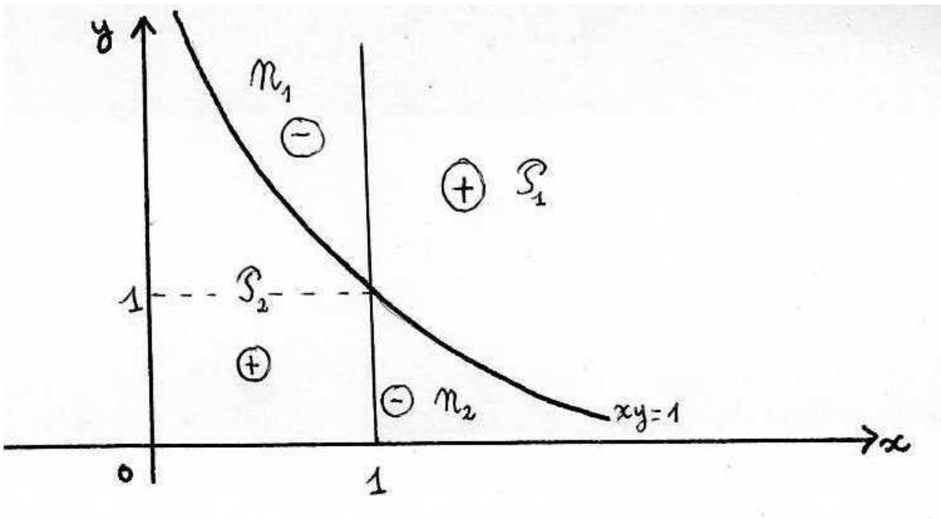

Figure 1

Let us define :

$$\psi(x,y) := l(x)l(y)\,\frac{x^2-1}{(xy-1)x}\,e^{-b(x+\frac{1}{x})} \tag{1.15}$$

It is clear that ψ is negative on $\mathcal{N}_1$ and $\mathcal{N}_2$ and positive on $\mathcal{P}_1$ and $\mathcal{P}_2$. We note :

$$N_i := \int\int_{\mathcal{N}_i} |\psi(x,y)|dx\,dy \qquad (i=1,2)$$
$$P_i := \int\int_{\mathcal{P}_i} \psi(x,y)\,dx\,dy \qquad (i=1,2)$$

• To prove (1.14) it suffices to see that : $N_i \le P_i$ $(i=1,2)$. To compute N_1 and P_2 $\big(\subset \{(x,y)\in\mathbb{R}_+^2;\ x\le 1\}\big)$, we make the change of variables for $x\in]0,1]$, $t\ge 2$: $x=\dfrac{t-\sqrt{t^2-4}}{2}$; so : $\dfrac{1}{x}=\dfrac{t+\sqrt{t^2-4}}{2}$, $x+\dfrac{1}{x}=t$, $\dfrac{x^2-1}{x^2}\,dx=dt$. We obtain :

$$\begin{aligned} N_1 &= \int_2^\infty dt\int_{\frac{t+\sqrt{t^2-4}}{2}}^\infty dy\, l\left(\frac{t-\sqrt{t^2-4}}{2}\right) l(y)\,\frac{e^{-bt}}{y-\frac{t+\sqrt{t^2-4}}{2}} \\ &= \int_2^\infty dt\int_0^\infty dz\, l\left(\frac{t-\sqrt{t^2-4}}{2}\right) l\left(\left(\frac{t+\sqrt{t^2-4}}{2}\right)(1+z)\right)\frac{e^{-bt}}{z} \end{aligned} \tag{1.16}$$

$\left(\text{after making the change of variable } y = (1+z)\left(\frac{t+\sqrt{t^2-4}}{2}\right)\right)$

$$P_2 = \int_2^{\infty} dt \int_0^1 dz\, l\left(\frac{t-\sqrt{t^2-4}}{2}\right) l\left(\left(\frac{t+\sqrt{t^2-4}}{2}\right)(1-z)\right)\frac{e^{-bt}}{z} \tag{1.17}$$

To compute N_2 and P_1 $\big(\subset \{(x,y)\in\mathbb{R}_+^2;\ x \geq\ 1\}\big)$ for $x \geq 1,\ t \geq 2$ we make the change of variable : $x = \dfrac{t+\sqrt{t^2-4}}{2}$. So $\dfrac{1}{x} = \dfrac{t-\sqrt{t^2-4}}{2}$, $x + \dfrac{1}{x} = t$ and $\dfrac{x^2-1}{x^2}\,dx = dt$. We obtain :

$$N_2 = \int_2^{\infty} dt \int_0^1 dz\, l\left(\frac{t+\sqrt{t^2-4}}{2}\right) l\left(\left(\frac{t-\sqrt{t^2-4}}{2}\right)(1-z)\right)\frac{e^{-bt}}{z} \tag{1.18}$$

$$P_1 = \int_2^{\infty} dt \int_0^{\infty} dz\, l\left(\frac{t+\sqrt{t^2-4}}{2}\right) l\left(\left(\frac{t-\sqrt{t^2-4}}{2}\right)(1+z)\right)\frac{e^{-bt}}{z} \tag{1.19}$$

We shall now use the hypothesis : $\underline{l \text{ belongs to } \mathcal{C}}$ to show that :

$$P_1 \geq N_1 \qquad \text{and} \qquad P_2 \geq N_2$$

which will end the proof of our Theorem.

• Comparing (1.19) and (1.16), it suffices, to prove that $P_1 \geq N_1$ to show that :

$$l\left(\frac{t+\sqrt{t^2-4}}{2}\right) l\left(\left(\frac{t-\sqrt{t^2-4}}{2}\right)(1+z)\right) \geq l\left(\frac{t-\sqrt{t^2-4}}{2}\right) l\left(\left(\frac{t+\sqrt{t^2-4}}{2}\right)(1+z)\right)$$

$$\text{i.e. :}\qquad l\left(\frac{1}{x}\right) l(c\,x) \geq l(x)\cdot l\left(\frac{c}{x}\right) \tag{1.20}$$

with $x \leq 1$ and $c \geq 1$.
If $a \geq 1$ (a being featured in the definition of $\mathcal{C}$), the relation (1.20) is trivially satisfied since $l(x) = 0$ for $x \leq a$ (and $x \leq 1$).
We now examine the case $0 \leq a < 1$.

If $x \leq a$, the relation (1.20) is again trivially satisfied. Thus, let us assume that $1 \geq x \geq a$. Relation (1.20) is equivalent to

$$\log l\left(\frac{1}{x}\right) - \log l(x) \geq \log l\left(\frac{c}{x}\right) - \log(c\,x)$$

or also to :

$$\int_x^{1/x} \frac{\theta(y)}{y}\,dy - \int_{cx}^{c/x} \frac{\theta(y)}{y}\,dy \geq 0 \tag{1.21}$$

$\left(\text{since } \log l(x) = \sigma + \displaystyle\int_b^x \frac{\theta(y)}{y}\,dy, \text{ from (1.1)}\right)$.
Thus, (1.21) is equivalent to :

$$\int_x^{1/x} \frac{\theta(y)}{y}\,dy - c\int_x^{1/x} \frac{\theta(cy)}{cy}\,dy = \int_x^{1/x} \frac{\theta(y)-\theta(cy)}{y}\,dy \geq 0 \tag{1.22}$$

and (1.22) is satisfied since θ is decreasing (and $c \geq 1$). We have shown that $P_1 \geq N_1$.
<u>We now show that $P_2 \geq N_2$</u> :
This time, using (1.17) and (1.18) it suffices to show that :

$$l\left(\frac{t-\sqrt{t^2-4}}{2}\right) l\left(\frac{t+\sqrt{t^2-4}}{2}(1-z)\right) \geq l\left(\frac{t+\sqrt{t^2-4}}{2}\right) l\left(\frac{t-\sqrt{t^2-4}}{2}(1-z)\right)$$

or, equivalently :

$$l(x)\, l\left(\frac{c}{x}\right) \geq l\left(\frac{1}{x}\right) l(cx) \qquad \text{with } x \leq 1 \text{ and } c \leq 1 \tag{1.23}$$

Relation (1.23) is trivial for $x \leq a$ $\big($since $cx \leq a$ and $l(cx) = 0\big)$. It remains to examine the case $x \geq a$, $a \leq 1$. Relation (1.23) is then equivalent to :

$$\int_{cx}^{\frac{c}{x}} \frac{\theta(y)}{y}\, dy - \int_{x}^{\frac{1}{x}} \frac{\theta(y)}{y}\, dy \geq 0, \quad \text{i.e.} \quad \int_{x}^{\frac{1}{x}} \frac{\theta(cy)-\theta(y)}{y}\, dy \geq 0.$$

The latter relation is obvious since θ is decreasing (and $c < 1$). This ends the proof of Theorem 2. ∎

Remark 3 Recall (see (1.8) above) that a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be hyperbolically completely monotone (HCM) if, for every $u > 0$, the function of w :

$$v + \frac{1}{v} = w \longrightarrow \varphi(uv)\varphi\left(\frac{u}{v}\right) \qquad (\text{with } v \geq 0)$$

is completely monotone. Thus, from (0.5), our Theorem 2 may be stated as follows : if l belongs to $\mathcal{C}$ then its Stieltjes transform is HCM.

2 Application to some r.v.'s related to recurrent linear diffusions

2.1 Our notation and hypotheses are now those of Salminen-Vallois-Yor $\big($[SVY]$\big)$ to which we refer the reader. $(X_t,\ t \geq 0)$ denotes a $\mathbb{R}_+$-valued diffusion which is recurrent ; we denote its speed measure (assumed to have no atoms) by m and its scale function by S. $(L_t,\ t \geq 0)$ denotes the (continuous) local time at 0 and $(\tau_u,\ u \geq 0)$ its right-continuous inverse :

$$\tau_u := \inf\{t \geq 0\ ;\ L_t > u\} \tag{2.1}$$

$(\tau_u,\ u \geq 0)$ is a subordinator whose Lévy measure admits a density $\big($see [SVY]$\big)$ which we shall denote by ν :

$$E(\exp -\lambda\tau_u) = \exp\left\{-u\int_0^\infty (1-e^{-\lambda x})\nu(x)dx\right\} \tag{2.2}$$

In fact, ν may be expressed in the form :

$$\nu(x) = \int_0^\infty e^{-xz} K(dz) \tag{2.3}$$

where K - the Krein measure - $\big($see Kotani-Watanabe [K.W], Knight [K]$\big)$ satisfies :

$$\int_0^\infty \frac{K(dz)}{z(1+z)} < \infty \qquad \text{and} \qquad \int_0^\infty \frac{K(dz)}{z} = \infty \tag{2.4}$$

2.2 Let, for every $t \geq 0$:

$$g_t := \sup\{s \leq t\ ;\ X_s = 0\}, \qquad d_t := \inf\{s \geq t\ ;\ X_s = 0\} \tag{2.5}$$

and denote by $\mathfrak{e}_p$ $(p > 0)$ an exponentially distributed variable with parameter p, i.e. with density $f_{\mathfrak{e}_p}(u) = p\, e^{-pu} 1_{u \geq 0}$; $\mathfrak{e}_p$ is assumed to be independent from $(X_t,\ t \geq 0)$. We define :

$$Y_p^{(1)} := \mathfrak{e}_p - g_{\mathfrak{e}_p}, \quad Y_p^{(2)} := d_{\mathfrak{e}_p} - \mathfrak{e}_p \quad Y_p^{(3)} := d_{\mathfrak{e}_p} - g_{\mathfrak{e}_p} \tag{2.6}$$

It is shown in [SVY], Theorem 16, that for $i = 1, 2, 3$, $Y_p^{(i)}$ is infinitely divisible.
More precisely, concerning $Y_p^{(3)}$, it is shown that $Y_p^{(3)}$ is a gamma-2 mixture, which implies from Kristiansen $\big($see [Kr]$\big)$ that $Y_p^{(3)}$ is infinitely divisible.
The aim of the following Theorem 4 is to improve, if possible, the results we just recalled. More precisely, we shall prove that, under certain hypotheses the r.v.'s $Y_p^{(2)}$ $(i = 1, 2, 3)$ are GGC r.v.'s whose Thorin measures have total masses $m \leq 1$. Thus, these variables :
- are GGC, hence are self-decomposable, and a fortiori are infinitely divisible,
- are gamma-m mixtures, with $m \leq 1$, and not only gamma-2 mixtures $\big($see (0.9)$\big)$.

Theorem 4 *We assume that Krein's measure K $\big($defined by (2.3)$\big)$ admits a differentiable density k.*
1. *Assume that :*

$$\frac{k'}{k}(x) = \frac{1}{x} + \frac{\theta(p+x)}{p+x}, \qquad \text{with } \theta \text{ decreasing}, \tag{2.7}$$

then $Y_p^{(1)}$ is a GGC r.v. whose Thorin measure is a subprobability.
2. *Assume that :*

$$\frac{k'}{k}(x) = \frac{1}{x+p} + \frac{\theta(x)}{x}, \qquad \text{with } \theta \text{ decreasing} \tag{2.8}$$

then $Y_p^{(2)}$ is a GGC r.v. whose Thorin measure is a subprobability.
3. *Assume that :*

$$\begin{cases} \dfrac{k'}{k}(z) = \dfrac{\theta(z)}{z} & \text{for } z < p \\[2ex] \dfrac{k'(z) - k'(z-p)}{k(z) - k(z-p)} = \dfrac{\theta(z)}{z} & \text{for } z \geq p \end{cases} \tag{2.9}$$

with θ decreasing

Then $Y_p^{(3)}$ is a GGC r.v. whose Thorin measure is a subprobability.

Proof of Theorem 4
We denote by $f_{Y_p^{(i)}}$ the density of $Y_p^{(i)}$. From [SVY], p. 115, we have :

$$f_{Y_p^{(1)}}(u) = C_1(p) \int_p^\infty e^{-uz} \cdot \frac{k(z-p)}{z-p} \, dz \tag{2.10}$$

$$f_{Y_p^{(2)}}(u) = C_2(p) \int_0^\infty e^{-uz} \, \frac{k(z)}{z+p} \, dz \tag{2.11}$$

$$f_{Y_p^{(3)}}(u) = C_3(p) \int_0^\infty e^{-uz} \big(k(z) - 1_{\{z \geq p\}} \, k(z-p)\big) dz \tag{2.12}$$

where $C_i(p)$, $i = 1, 2, 3$ are three normalising constants. We shall now use Theorem 2 with, successively :

$$l^{(1)}(x) = C_1(p) \, \frac{k(x-p)}{x-p} \, 1_{x \geq p} \tag{2.13}$$

$$l^{(2)}(x) = C_2(p) \, \frac{k(x)}{x+p} \tag{2.14}$$

$$l^{(3)}(x) = C_3(p) \big(k(x) - 1_{x \geq p} \, k(x-p)\big) \tag{2.15}$$

We already note that, for $i = 1, 2, 3$, $\displaystyle\int_0^\infty \frac{l^{(i)}(x)}{x} \, dx < \infty$. Indeed :

$$\begin{aligned}
\int_0^\infty \frac{l^{(1)}(x)}{x} \, dx &= C_1(p) \int_p^\infty \frac{k(x-p)}{x(x-p)} \, dx = C_1(p) \int_0^\infty \frac{k(x)}{x(x+p)} \, dx \\
&< \infty \quad \big(\text{from (2.4)}\big) \\
\int_0^\infty \frac{l^{(2)}(x)}{x} \, dx &= C_2(p) \int_0^\infty \frac{k(x)}{x(x+p)} \, dx < \infty \quad \big(\text{from (2.4)}\big) \\
\int_0^\infty \frac{l^{(3)}(x)}{x} \, dx &= C_3(p) \int_0^\infty k(x) \left(\frac{1}{x} - \frac{1}{x+p}\right) dx \\
&= p \, C_3(p) \int_0^\infty \frac{k(x)}{x(x+p)} \, dx < \infty \quad \big(\text{from 2.4)}\big)
\end{aligned}$$

Finally, it remains to observe that hypothesis (2.7) $\big($resp. (2.8), resp. (2.9)$\big)$ implies that $l^{(1)} \in \mathcal{C}$ (resp. $l^{(2)} \in \mathcal{C}$, resp. $l^{(3)} \in \mathcal{C}$). ■

3 Application to recurrent Bessel processes

3.1 The notation is the same as in the preceding part, but, now $(X_t,\ t \geq 0)$ is a Bessel process with dimension $d = 2(1-\alpha)$ with $0 < d < 2$, or equivalently $0 < \alpha < 1$.

Theorem 5 *For any $\alpha \in]0,1[$, for any $p > 0$, the r.v.'s*

$$Y_p^{(1)} = \mathfrak{e}_p - g_{\mathfrak{e}_p}, \quad Y_p^{(2)} = d_{\mathfrak{e}_p} - \mathfrak{e}_p, \quad Y_p^{(3)} = d_{\mathfrak{e}_p} - g_{\mathfrak{e}_p}$$

are GGC r.v.'s whose Thorin measures have the same total mass : $1 - \alpha = \dfrac{d}{2}\,(< 1)$.

Proof of Theorem 5

We already note that, since :

$$\nu(a) = \int_0^\infty e^{-az}\, K(dz) \qquad \big(\text{from (2.3)}\big)$$

and $$\nu(a) = \frac{1}{2^\alpha \Gamma(\alpha)}\, \frac{1}{a^{\alpha+1}} \qquad \big(\text{from [D-M, RVY], p. 5}\big)$$

then the density k of Krein's measure equals here :

$$k(a) = \frac{1}{2^\alpha \Gamma(\alpha)\Gamma(\alpha+1)}\, a^\alpha \qquad (a>0) \tag{3.1}$$

1. We begin by proving Theorem 5 for the r.v. $Y^{(2)}$.

(To simplify the notation, we write $Y^{(2)}$ instead $Y_p^{(2)}$). To see that $Y^{(2)}$ is GGC, it suffices, from Theorem 2, to show that $l^{(2)} \in \mathcal{C}$ where here :

$$l^{(2)}(x) = C\, \frac{x^\alpha}{x+p} \qquad \big(\text{from (3.1) and (2.14)}\big) \tag{3.2}$$

Thus :

$$x\big(\log\, l^{(2)}\big)'(x) = \alpha - \frac{x}{x+p} = \alpha - 1 + \frac{p}{x+p}$$

is a decreasing function of x, hence $l^{(2)} \in \mathcal{C}$ from (1.2). It remains to see that the total mass of the Thorin measure of $Y^{(2)}$ equals $1-\alpha$. Now, from (0.10), this total mass m equals :

$$\begin{aligned} m \;&:=\; \sup\left\{\delta \geq 0\;;\; \lim_{u\downarrow 0_+} \frac{1}{u^{\delta-1}}\, f_{Y^{(2)}}(u) = 0\right\} \\ &=\; \sup\left\{\delta \geq 0\;;\; \lim_{u\downarrow 0_+} \frac{C}{u^{\delta-1}} \int_0^\infty e^{-ux}\, \frac{x^\alpha}{x+p}\, dx = 0\right\} \end{aligned} \tag{3.3}$$

However, since the function $x \to \dfrac{x^\alpha}{x+p}$ decreases for x large enough and is equivalent to $x^{\alpha-1}$ when $x\to\infty$, the Tauberian Theorem implies :

$$f_{Y^{(2)}}(u) \underset{u\to 0}{\sim} \frac{C'}{u^\alpha} \tag{3.4}$$

It is then clear that (3.3) and (3.4) imply $m = 1-\alpha$.

2. We now prove Theorem 5 for the r.v. $Y^{(1)}$.

For this purpose, we shall use a more direct method than relying on Theorem 2. Indeed, we have, from (0.5), (2.13) and (3.1) :

$$\begin{aligned} E(e^{-\lambda Y^{(1)}}) \;&=\; \int_0^\infty \frac{l^{(1)}(z)}{\lambda+z}\, dz = C\int_p^\infty \frac{1}{\lambda+z}(z-p)^{\alpha-1} dz \\ &=\; C\int_0^\infty \frac{1}{\lambda+p+z}\, z^{\alpha-1} dz = C\int_0^\infty z^{\alpha-1} dz \int_0^\infty e^{-(\lambda+p+z)u} du \\ &=\; C\int_0^\infty e^{-(\lambda+p)u} du \int_0^\infty e^{-zu} z^{\alpha-1} dz \\ &=\; C\Gamma(\alpha)\int_0^\infty e^{-(\lambda+p)u}\, \frac{du}{u^\alpha} = (\lambda+p)^{\alpha-1} C\Gamma(\alpha)\Gamma(1-\alpha) \\ &=\; \left(1+\frac{\lambda}{p}\right)^{\alpha-1} \end{aligned} \tag{3.5}$$

since the Laplace transform $E(e^{-\lambda Y^{(1)}})$ equals 1 for $\lambda = 0$. Thus :

$$Y^{(1)} \overset{\text{(law)}}{=} \frac{1}{p}\,\gamma_{1-\alpha} \text{ where } \gamma_{1-\alpha} \text{ is a gamma r.v. with parameter } 1-\alpha, \text{ i.e.} \tag{3.6}$$

with density :

$$f_{\gamma_{1-\alpha}}(u) := \frac{e^{-u}}{\Gamma(1-\alpha)}\, u^{-\alpha}\, 1_{u\geq 0}$$

It follows clearly from (3.5) that :

$$\begin{aligned} E(e^{-\lambda Y^{(1)}}) &= \exp\left\{-(1-\alpha)\,\log\left(1+\frac{\lambda}{p}\right)\right\} \\ &= \exp\left\{-(1-\alpha)\int_0^\infty (1-e^{-\lambda x})\,\frac{dx}{x}\,e^{-xp}\right\} \end{aligned} \tag{3.7}$$

Thus, from (0.7), formula (3.7) shows that $Y^{(1)}$ is a GGC variable with Thorin measure $(1-\alpha)\delta_p$.

3. We now prove Theorem 5 for the r.v. $Y_p^{(3)}$.
In fact, this result - $Y^{(3)}$ is a GGC variable whose Thorin measure has total mass equal to $1-\alpha$ - has already been proven in [BFRY] (with $p = 1$, but this involves no loss of generality). The proof we shall give now is a totally different one from that of [BFRY]. We also assume here, for simplicity, that $p = 1$ and we denote $Y^{(3)}$ instead of $Y_1^{(3)}$.
Following the arguments of the proof of Theorem 2, we need to show, from (1.16), (1.17), (1.18) and (1.19) that, for every $x \in [0,1]$:

$$\begin{aligned} \Delta(x) &= \int_0^\infty \left\{ l\left(\frac{1}{x}\right) l\big(x(1+z)\big) - l(x) l\left(\frac{1}{x}(1+z)\right)\right\}\frac{dz}{z} \\ &+ \int_0^1 \left\{ l(x) l\left(\frac{1}{x}(1-z)\right) - l\left(\frac{1}{x}\right) l\big(x(1-z)\big)\right\}\frac{dz}{z} \geq 0 \end{aligned} \tag{3.8}$$

where the function $l (= l^{(3)})$ equals here, from (3.1) and (2.15) :

$$l(y) = y^\alpha - 1_{y\geq 1}\,(y-1)^\alpha \qquad (y \geq 0) \tag{3.9}$$

Thus, we need to show (3.8). For this purpose, we need to compute the integrals featured in (3.8) hence, given (3.9) to discuss, owing to the positions of $x(1+z)$, $\frac{1}{x}(1+z)$, $\frac{1}{x}(1-z)$ and $x(1-z)$ with respect to 1. We consider the first integral in (3.8) for $x(1+z) \geq 1$ $\Big($hence, a fortiori $\frac{1}{x}(1+z) \geq 1$ since $x \leq 1\Big)$. This first term equals :

$$\begin{aligned}
&\Delta_1(x) \\
&= \int_{\frac{1}{x}-1}^{\infty} \left\{ \left(\frac{1}{x^\alpha} - \left(\frac{1}{x}-1\right)^\alpha \right) \left(x^\alpha (1+z)^\alpha - \big(x(1+z)-1\big)^\alpha \right) \right. \\
&\qquad\qquad \left. -x^\alpha \left[\left(\frac{1}{x^\alpha}(1+z)^\alpha \right) - \left(\frac{1}{x}(1+z) - 1 \right)^\alpha \right] \right\} \frac{dz}{z} \\
&= \int_{\frac{1}{x}-1}^{\infty} \left(1-(1-x)^\alpha\right) \left((1+z)^\alpha - \left(1+z-\frac{1}{x}\right)^\alpha \right) - \left((1+z)^\alpha - (1+z-x)^\alpha\right) \frac{dz}{z} \\
&= \int_{\frac{1}{x}-1}^{\infty} \left\{ \left[(1+z-x)^\alpha - \left(1+z-\frac{1}{x}\right)^\alpha \right] - \left[(1-x)^\alpha (1+z)^\alpha - \left((1-x)(1+z) - \frac{1-x}{x} \right)^\alpha \right] \right\} \frac{dz}{z} \\
&:= \Delta_1^{(1)}(x) - \Delta_1^{(2)}(x).
\end{aligned}$$

Let us examine $\Delta_1^{(1)}(x)$:

$$\begin{aligned}
\Delta_1^{(1)}(x) &= \int_{\frac{1}{x}-1}^{\infty} \left[(1+z-x)^\alpha - \left(1+z-\frac{1}{x}\right)^\alpha \right] \frac{dz}{z} \\
&= \int_{\frac{1}{x}-1}^{\infty} \frac{dz}{z} \int_{1+z-\frac{1}{x}}^{1+z-x} \alpha\, u^{\alpha-1} du
\end{aligned}$$

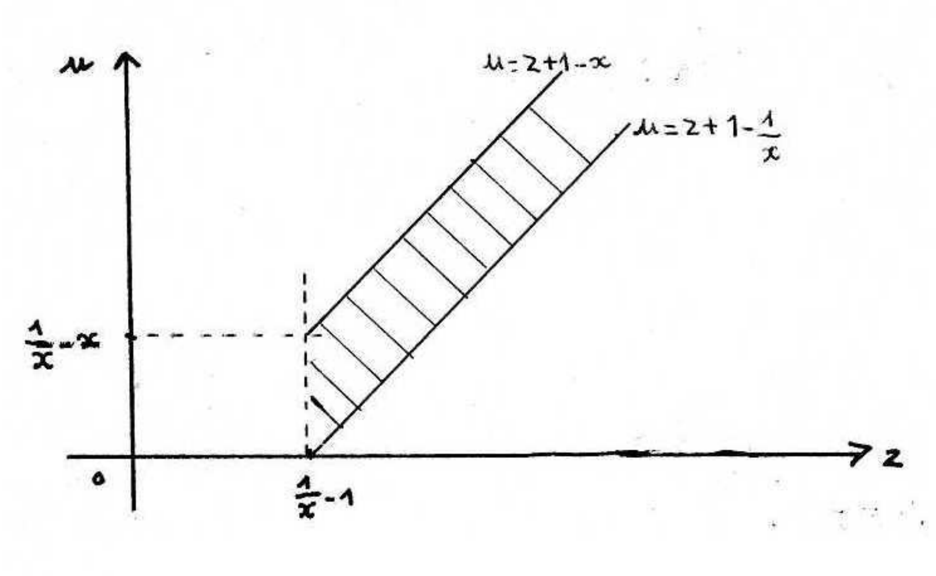

Figure 2

Now, we apply Fubini's Theorem :

$$\begin{aligned}
\Delta_1^{(1)}(x) &= \int_0^{\frac{1}{x}-1} \alpha\, u^{\alpha-1} du \int_{\frac{1}{x}-1}^{u+\frac{1}{x}-1} \frac{dz}{z} + \int_{\frac{1}{x}-1}^{\infty} \alpha\, u^{\alpha-1} du \int_{u+x-1}^{u+\frac{1}{x}-1} \frac{dz}{z} \\
&= \int_0^{\frac{1}{x}-1} \alpha\, u^{\alpha-1} \log\left(\frac{ux+1-x}{1-x} \right) du + \int_{\frac{1}{x}-1}^{\infty} \alpha\, u^{\alpha-1} \log\left(\frac{u+\frac{1}{x}-1}{u+x-1} \right) du.
\end{aligned}$$

We compute thus each term of $\Delta(x)$ and we obtain, after some simple, although tedious, computations :

$$\begin{aligned}
\Delta(x) &= \int_0^{\frac{1}{x}-1} \log\left(\frac{1}{1-x}\right)\cdot \alpha\, u^{\alpha-1}du + \int_0^{\frac{1}{x}-x} \log\left(\frac{ux+1-x}{1-x}\right)\alpha\, u^{\alpha-1}du \\
&\quad + \int_{\frac{1}{x}-x}^{\infty} \log\left(\frac{u+\frac{1}{x}-1}{u+x-1}\right)\alpha\, u^{\alpha-1}du + \int_0^{x(1-x)} \log\left(\frac{1-x-u}{(1-x)^2}\right)\alpha\, u^{\alpha-1}du \\
&\quad - \int_0^{\frac{1}{x}-1} \log\left(\frac{\frac{u}{1-x}+\frac{1}{x}-1}{\frac{1}{x}-1}\right)\alpha\, u^{\alpha-1}du - \int_{\frac{1}{x}-1}^{\infty} \log\left(\frac{\frac{u}{1-x}+\frac{1}{x}-1}{\frac{u}{1-x}-1}\right)\alpha\, u^{\alpha-1}du \quad (3.10)
\end{aligned}$$

We note that, since $x \in [0,1]$, we have :

$$x(1-x) \le 1-x \le \frac{1}{x}-1 \le \frac{1}{x}-x$$

and that all the integrals found in (3.10) are positive. In (3.10) we shall gather the terms with opposite signs. For example, we have :

$$\begin{aligned}
&\int_{\frac{1}{x}-x}^{\infty} \log\left(\frac{u+\frac{1}{x}-1}{u+x-1}\right)\alpha\, u^{\alpha-1}du - \int_{\frac{1}{x}-x}^{\infty} \log\left(\frac{\frac{u}{1-x}+\frac{1}{x}-1}{\frac{u}{1-x}-1}\right)\alpha\, u^{\alpha-1}du \\
&= \int_{\frac{1}{x}-x}^{\infty} \log\left(\frac{ux+(1-x)}{ux+(1-x)^2}\right)\alpha\, u^{\alpha-1}du
\end{aligned}$$

and this last integral is positive since $(1-x)^2 \le 1-x$. Gathering thus all the terms in $\Delta(x)$, we obtain :

$$\begin{aligned}
\Delta(x) &= \int_0^{\infty} \log\left(\frac{ux+1-x}{ux+(1-x)^2}\right)\alpha\, u^{\alpha-1}du - \int_{\frac{1}{x}-1}^{\frac{1}{x}-x} \log\left(\frac{1-x}{x(u+x-1)}\right)\alpha\, u^{\alpha-1}du \\
&\qquad \int_0^{x(1-x)} \log\left(\frac{1-x-u}{(1-x)^2}\right)\alpha\, u^{\alpha-1}du \qquad (3.11) \\
&= \alpha(1-x)^{\alpha} \int_{\frac{1}{x}}^{\frac{1}{x}+1} v^{\alpha-1}\log\left(\frac{1}{x(v-1)}\right) \\
&\qquad \left[\left(\frac{xv-1}{1+v-xv}\right)^{\alpha-1}\frac{x^{2-\alpha}}{(1+x-xv)^2} + \left(\frac{xv-1}{x(v-1)}\right)^{\alpha-1}\frac{1-x}{x(v-1)^2} - v^{\alpha-1}\right]dv
\end{aligned}$$

after making the changes of variables :

$\dfrac{ux+1-x}{ux+(1-x)^2} = \dfrac{1}{x(v-1)}$	in the first integral of (3.11)
$u=(1-x)v$	in the second integral of (3.11)
$\dfrac{1-x-u}{(1-x)^2} = \dfrac{1}{x(v-1)}$	in the third integral of (3.11)

Thus, to conclude, it remains to show that, for every $v \in \left[\frac{1}{x}, \frac{1}{x}+1\right]$:

$$v^{\alpha-1} \le \left(\frac{xv-1}{1+x-xv}\right)^{\alpha-1}\frac{1}{x^{\alpha-1}}\frac{x}{(1+x-xv)^2} + \left(\frac{xv-1}{x(v-1)}\right)^{\alpha-1}\frac{1-x}{x(v-1)^2} \qquad (3.12)$$

or, equivalently that :

$$\left(\frac{xv-1}{xv}\right)^{1-\alpha} \leq \frac{x}{(1+x-xv)^{\alpha+1}} + \frac{1-x}{x}\,\frac{1}{(v-1)^{\alpha+1}} \tag{3.13}$$

Now, this last inequality is obvious ; indeed, since $f_1(v) := \left(\frac{xv-1}{xv}\right)^{1-\alpha}$ is increasing as well as $f_2(v) := \frac{x}{(1+x-xv)^{\alpha+1}}$ it suffices to verify that $f_1\left(\frac{1}{x}+1\right) \leq f_2\left(\frac{1}{x}\right)\cdot$ We have :

$$f_1\left(\frac{1}{x}+1\right) = \left(\frac{x}{1+x}\right)^{1-\alpha} \leq 1 \leq f_2\left(\frac{1}{x}\right) = \frac{1}{x^\alpha}$$

since $x \in [0,1]$. This shows that $Y^{(3)}$ is GGC. Finally, it is not difficult to prove that the total mass of the Thorin measure equals $1-\alpha$: this follows from the fact that since $l^{(3)}(x) = C\big(x^\alpha - 1_{x\geq 1}(x-1)^\alpha\big)$ then $l^{(3)}(x) \underset{x\to\infty}{\sim} C\,x^{\alpha-1}$, hence, from the Tauberian Theorem, $f_{Y^{(3)}}(u) \underset{u\to 0}{\sim} \frac{C}{u^\alpha}$, and we finally use (0.10). ■

3.2 Description of the r.v.'s $\mathbb{G}_\alpha^{(i)}$ $(i=1,2,3\ ;\ 0<\alpha<1)$

In the sequel, it will convenient to assume that $p=1$ and we write simply $Y^{(i)}$ for the r.v.'s $Y_1^{(1)}$ $(i=1,2,3)$. Theorem 5 implies, from (0.7), the existence of r.v.'s

$$\begin{aligned}&\mathbb{G}_\alpha^{(i)} \quad \big(i=1,2,3\ ;\ \alpha\in]0,1[\big) \text{ such that } \quad E\big(\log^+(1/\mathbb{G}_\alpha^{(i)})\big)<\infty \quad \text{and :}\\ &E(e^{-\lambda Y^{(i)}}) = \exp\left\{-(1-\alpha)\int_0^\infty (1-e^{-\lambda x})\frac{dx}{x}\,E(e^{-x\mathbb{G}_\alpha^{(i)}})\right\}\end{aligned} \tag{3.14}$$

The aim of this section is to identify the (laws of the) r.v.'s $\mathbb{G}_\alpha^{(i)}$ and to describe some of their properties.

i) The case $i=1$

Formula (3.6) implies that the r.v. $\mathbb{G}_\alpha^{(1)}$ is a.s. equal to 1, i.e. its distribution is δ_1, the Dirac measure at 1. In particular, this distribution does not depend on α.

ii) The case $i=3$

In [BFRY] a complete study of the r.v.'s $\mathbb{G}_\alpha^{(3)}$ - denoted as $\mathbb{G}_\alpha$ in [BFRY] - has been undertaken. We refer the reader to [BFRY]. In particular, it is shown there that the density $f_{\mathbb{G}_\alpha^{(3)}}$ of $\mathbb{G}_\alpha^{(3)}$ equals :

$$f_{\mathbb{G}_\alpha^{(3)}}(u) = \frac{\alpha\sin\pi\alpha}{(1-\alpha)\pi}\,\frac{u^{\alpha-1}(1-u)^{\alpha-1}}{(1-u)^{2\alpha}-2(1-u)^\alpha u^\alpha\cos(\pi\alpha)+1}\,1_{[0,1]}(u) \tag{3.15}$$

Thus, $\mathbb{G}_{1/2}^{(3)}$ is arc-sine distributed :

$$f_{\mathbb{G}_{1/2}^{(3)}}(u) = \frac{1}{\pi}\,\frac{1}{\sqrt{u(1-u)}}\,1_{[0,1]}(u) \tag{3.16}$$

and the r.v.'s $\mathbb{G}_\alpha^{(3)}$ converge in law, as $\alpha \to 0$ and $\alpha \to 1$ respectively towards $\mathbb{G}_0^{(3)}$ and $\mathbb{G}_1^{(3)}$, where :

$$\mathbb{G}_0^{(3)} \overset{(\mathrm{law})}{=} \frac{1}{1+\exp \pi C}, \quad \text{with } C \text{ a standard Cauchy r.v.} \tag{3.17}$$

$$\mathbb{G}_1^{(3)} \overset{(\mathrm{law})}{=} U, \quad \text{with } U \text{ uniform on } [0,1] \tag{3.18}$$

iii) The case $i = 2$

Theorem 6 *For every $\alpha \in]0,1[$*

$$1) \quad i) \quad Y^{(2)} \overset{(\mathrm{law})}{=} \mathfrak{e} \cdot \frac{\gamma_{1-\alpha}}{\gamma_\alpha} \overset{(\mathrm{law})}{=} \mathfrak{e} \, \frac{\beta_{1-\alpha,\alpha}}{1-\beta_{1-\alpha,\alpha}} \tag{3.19}$$

where $\mathfrak{e}, \gamma_{1-\alpha}, \gamma_\alpha$ are independent, with respective laws the standard exponential and the gamma distributions with respective parameters $(1-\alpha)$ and α, and where $\mathfrak{e}$ and $\beta_{1-\alpha,\alpha}$ are independent with respective distributions the standard exponential and the beta distribution with parameters $(1-\alpha,\alpha)$.

$$ii) \quad E(e^{-\lambda Y^{(2)}}) = \frac{\lambda^\alpha - 1}{\lambda - 1} \quad (= \alpha \text{ if } \lambda = 1) \quad (\lambda \geq 0) \tag{3.20}$$

2) *$Y^{(2)}$ is a gamma-$(1-\alpha)$ mixture, i.e. :*

$$Y^{(2)} = \gamma_{1-\alpha} \cdot D_{1-\alpha}^{(2)} \tag{3.21}$$

where $\gamma_{1-\alpha}$ is a gamma $(1-\alpha)$ variable, independent from the positive r.v. $D_{1-\alpha}^{(2)}$. Furthermore :

$$D_{1-\alpha}^{(2)} \overset{\mathrm{law}}{=} \frac{\mathfrak{e}}{\gamma_\alpha} \tag{3.22}$$

$$E(e^{-\lambda D_{1-\alpha}^{(2)}}) = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-y} \frac{y^\alpha}{\lambda + y} \, dy = \alpha \int_0^\infty \frac{e^{-\lambda y}}{(1+y)^{\alpha+1}} \, dy \tag{3.23}$$

The density $f_{D_{1-\alpha}^{(2)}}$ of $D_{1-\alpha}^{(2)}$ equals :

$$f_{D_{1-\alpha}^{(2)}}(u) = \frac{\alpha}{(1+u)^{\alpha+1}} \, 1_{[0,\infty[}(u) \tag{3.24}$$

3) *i) The density $f_{\mathbb{G}_\alpha^{(2)}}$ of $\mathbb{G}_\alpha^{(2)}$ equals :*

$$f_{\mathbb{G}_\alpha^{(2)}}(u) = \frac{\alpha \sin(\pi\alpha)}{(1-\alpha)\pi} \, \frac{u^{\alpha-1}}{u^{2\alpha} - 2u^\alpha \cos(\pi\alpha) + 1} \, 1_{[0,\infty[}(u) \tag{3.25}$$

ii) The r.v.'s $\mathbb{G}_\alpha^{(2)}$ are related to the r.v.'s $\mathbb{G}_\alpha^{(3)}$ via the identity is law :

$$\frac{\mathbb{G}_\alpha^{(2)}}{1+\mathbb{G}_\alpha^{(2)}} \overset{(\mathrm{law})}{=} \mathbb{G}_\alpha^{(3)}, \text{ or, equivalently, } \quad \mathbb{G}_\alpha^{(2)} \overset{(\mathrm{law})}{=} \frac{\mathbb{G}_\alpha^{(3)}}{1-\mathbb{G}_\alpha^{(3)}} \tag{3.26}$$

$$iii) \qquad \mathbb{G}_\alpha^{(2)} \overset{(\mathrm{law})}{=} \frac{1}{\mathbb{G}_\alpha^{(2)}} \tag{3.27}$$

iv) As $\alpha \to 0$ and $\alpha \to 1$, $\mathbb{G}_\alpha^{(2)}$ converges in law towards, respectively

$$\mathbb{G}_0^{(2)} \overset{\text{(law)}}{=} \exp \pi\, C \qquad \text{and} \qquad \mathbb{G}_{\mathbf{1}}^{(2)} \overset{\text{(law)}}{=} \frac{U}{1-U} \tag{3.28}$$

with C a standard Cauchy r.v. and U uniform on $[0,1]$.

4) *Let $\mu \in]0,1[$ and T_μ denote the positive stable r.v. with index μ whose law is characterized by :*

$$E(e^{-\lambda T_\mu}) = \exp(-\lambda^\mu) \qquad (\lambda > 0)$$

Then :

$$\mathbb{G}_\alpha^{(2)} \overset{\text{(law)}}{=} \left(\frac{T_{1-\alpha}}{T'_{1-\alpha}}\right)^{\frac{1-\alpha}{\alpha}} \tag{3.29}$$

where $T'_{1-\alpha}$ is an independent copy of $T_{1-\alpha}$.

ii) An equivalent way of writing (3.29) is :

$$\mathbb{G}_\alpha^{(2)} \overset{\text{(law)}}{=} \left(\frac{M_{1-\alpha}}{M'_{1-\alpha}}\right)^{\frac{1}{\alpha}} \tag{3.30}$$

where $M_{1-\alpha}$ and $M'_{1-\alpha}$ are two independent Mittag-Leffler r.v.'s with parameter $1-\alpha$, whose common law is characterized by :

$$\begin{aligned} E(\exp \lambda\, M_{1-\alpha}) &= \sum_{n\geq 0} \frac{\lambda^n}{\Gamma\big(1+n(1-\alpha)\big)}, \ E[M_{1-\alpha}^n] = \frac{\Gamma(n+1)}{\Gamma\big(1+n(1-\alpha)\big)} \\ M_{1-\alpha} &\overset{\text{(law)}}{=} \left(\frac{1}{T_{1-\alpha}}\right)^{1-\alpha} \end{aligned} \tag{3.31}$$

$\big($*see [CY], p.114, Exercise 4.19* $\big)$.

Proof of Theorem 6

1) *i)* We prove (3.19)

Denoting by $(R_t,\ t \geq 0)$ a Bessel process with dimension $2(1-\alpha)$ $(0<\alpha<1)$ starting from 0, we have by scaling :

$$Y^{(2)} = d_{\mathfrak{e}} - \mathfrak{e} \overset{\text{(law)}}{=} \mathfrak{e}(d_1 - 1) \overset{\text{(law)}}{=} \mathfrak{e}\left(\frac{R_1^2}{2\gamma_\alpha}\right)$$

$\big($see [BFRY]$\big)$ where R_1^2 is the value of R_t^2 for $t=1$. Hence :

$$Y^{(2)} \overset{\text{(law)}}{=} \mathfrak{e}\ \frac{\gamma_{1-\alpha}}{\gamma_\alpha} = \mathfrak{e}\ \frac{\beta_{1-\alpha,\,\alpha}}{1-\beta_{1-\alpha,\,\alpha}}$$

(from the classical "beta-gamma algebra").

ii) We prove (3.20)

We have, from (3.1) and (3.2) :

$$l^{(2)}(x) = \frac{\sin(\pi\alpha)}{\pi}\ \frac{x^\alpha}{1+x} \qquad x \geq 0$$

$\Big($we note that $\displaystyle\int_0^\infty \frac{l^{(2)}(x)}{x}\,dx = \frac{\sin(\pi\alpha)}{\pi}\int_0^\infty \frac{x^{\alpha-1}}{1+x}\,dx = \frac{\sin(\pi\alpha)}{\pi}\,B(\alpha,1-\alpha) = \frac{\sin(\pi\alpha)}{\pi}\,\Gamma(\alpha)$ $\Gamma(1-\alpha) = 1$ (see [L], p. 3 and 13)$\Big)$. Hence from (0.3), $f_{Y^{(2)}}$, the density of $Y^{(2)}$, equals :

$$f_{Y^{(2)}}(u) = \frac{\sin(\pi\alpha)}{\pi}\int_0^\infty e^{-ux}\frac{x^\alpha}{1+x}\,dx$$

(we might also have derived this formula from (3.19)).

iii) We now compute the Laplace transform of $Y^{(2)}$

$$\begin{aligned}
E(e^{-\lambda Y^{(2)}}) &= \frac{\sin(\pi\alpha)}{\pi}\int_0^\infty e^{-\lambda u}du\int_0^\infty e^{-ux}\frac{x^\alpha}{1+x}\,dx \\
&= \frac{\sin(\pi\alpha)}{\pi}\int_0^\infty \frac{x^\alpha}{(1+x)(\lambda+x)}\,dx \\
&= \frac{1}{\lambda-1}\,\frac{\sin(\pi\alpha)}{\pi}\int_0^\infty x^\alpha\left[\frac{1}{1+x}-\frac{1}{\lambda+x}\right]dx \\
&= \lim_{A\to\infty}\frac{1}{\lambda-1}\,\frac{\sin(\pi\alpha)}{\pi}\left[\int_0^A \frac{x^\alpha}{1+x}\,dx - \lambda^\alpha\int_0^{\frac{A}{\lambda}}\frac{x^\alpha}{1+x}\,dx\right] \\
&= \lim_{A\to\infty}\frac{1}{\lambda-1}\,\frac{\sin(\pi\alpha)}{\pi}\left[\int_0^A\left(x^{\alpha-1}-\frac{x^{\alpha-1}}{1+x}\right)dx - \lambda^\alpha\int_0^{\frac{A}{\lambda}}\left(x^{\alpha-1}-\frac{x^{\alpha-1}}{1+x}\right)dx\right] \\
&= \lim_{A\to\infty}\frac{1}{\lambda-1}\,\frac{\sin(\pi\alpha)}{\pi}\left[\frac{A^\alpha}{\alpha}-\int_0^\infty\frac{x^{\alpha-1}}{1+x}\,dx-\frac{\lambda^\alpha}{\alpha}\left(\frac{A}{\lambda}\right)^\alpha+\lambda^\alpha\int_0^\infty\frac{x^{\alpha-1}}{1+x}\,dx\right] \\
&= \frac{\lambda^\alpha-1}{\lambda-1}\,\frac{\sin(\pi\alpha)}{\pi}\int_0^\infty\frac{x^{\alpha-1}}{1+x}\,dx = \frac{\lambda^\alpha-1}{\lambda-1}\,\frac{\sin(\pi\alpha)}{\pi}\,B(\alpha,1-\alpha) \\
&= \frac{\lambda^\alpha-1}{\lambda-1}
\end{aligned}$$

$\Big($since (see [L], p. 3) $\quad B(\alpha,1-\alpha) = \Gamma(\alpha)\Gamma(1-\alpha) = \dfrac{\pi}{\sin(\pi\alpha)}\Big)$.

2) Let us show (3.25)

By taking the logarithmic derivative of (3.20) :

$$E(e^{-\lambda Y^{(2)}}) = \frac{\lambda^\alpha-1}{\lambda-1} = \exp\left\{-(1-\alpha)\int_0^\infty(1-e^{-\lambda x})\frac{dx}{x}\,E(e^{-x\,\mathbb{G}_\alpha^{(2)}})\right\}$$

we obtain :

$$E\left[\frac{1}{\lambda+\mathbb{G}_\alpha^{(2)}}\right] = \frac{1}{1-\alpha}\,\left[\frac{1}{1-\lambda}-\frac{\alpha\lambda^{\alpha-1}}{\lambda^\alpha-1}\right] \tag{3.32}$$

Thus, we have just computed the Stieltjes transform of the r.v. $\mathbb{G}_\alpha^{(2)}$. The inversion formula for the Stieltjes transform (see [W], p. 345) leads us to :

$$\begin{aligned} f_{\mathbb{G}_\alpha^{(2)}}(u) &= \frac{1}{2i\pi(1-\alpha)} \lim_{\eta\to 0} \left[\frac{1}{1-\lambda(-u-i\eta)} - \frac{\alpha(-u-i\eta)^{\alpha-1}}{(-u-i\eta)^\alpha - 1} - \frac{1}{1-\lambda(-u+i\eta)} \right. \\ &\qquad \left. + \frac{\alpha(-u+i\eta)^{\alpha-1}}{(-u+i\eta)^\alpha-1} \right] \qquad (u>0) \\ &= \frac{-\alpha}{2i\pi(1-\alpha)} \left[\frac{-u^{\alpha-1}\, e^{-i\pi\alpha}}{u^\alpha\, e^{-i\pi\alpha}-1} + \frac{u^{\alpha-1}\, e^{i\pi\alpha}}{u^\alpha\, e^{i\pi\alpha}-1} \right] \qquad (u>0) \end{aligned}$$

(We note that, in the preceding limit, the contribution of the term $\dfrac{1}{1-\lambda}$ is 0).

$$\begin{aligned} &= \frac{-\alpha}{2i\pi(1-\alpha)} \left[\frac{-u^{2\alpha-1} + u^{\alpha-1}\, e^{-i\pi\alpha} + u^{2\alpha-1} - u^{\alpha-1}\, e^{i\pi\alpha}}{u^{2\alpha} - u^\alpha\, e^{i\pi\alpha} - u^\alpha\, e^{-i\pi\alpha} + 1} \right] \qquad (u>0) \\ &= \frac{\alpha\sin(\pi\alpha)}{(1-\alpha)\pi} \, \frac{u^{\alpha-1}}{u^{2\alpha} - 2u^\alpha\cos(\pi\alpha)+1}\, 1_{(u>0)} \end{aligned}$$

3) We now show (3.26)
For every h Borel and positive, we have :

$$E\left[h\left(\frac{\mathbb{G}_\alpha^{(2)}}{1+\mathbb{G}_\alpha^{(2)}} \right) \right] = \frac{\alpha\sin(\pi\alpha)}{(1-\alpha)\pi} \int_0^\infty h\left(\frac{u}{1+u}\right) \frac{u^{\alpha-1}}{u^{2\alpha}-2u^\alpha\cos(\pi\alpha)+1}\, du$$
(from (3.25))

Thus, making the change of variable $\dfrac{u}{1+u} = x$:

$$\begin{aligned} E\left[h\left(\frac{\mathbb{G}_\alpha^{(2)}}{1+\mathbb{G}_\alpha^{(2)}} \right) \right] &= \frac{\alpha\sin(\pi\alpha)}{(1-\alpha)\pi} \int_0^1 h(x) \frac{dx}{(1-x)^2} \, \frac{\frac{x^{\alpha-1}}{(1-x)^{\alpha-1}}}{\frac{x^{2\alpha}}{(1-x)^{2\alpha}} - \frac{2\cos(\pi\alpha)x^\alpha}{(1-x)^\alpha} + 1} \\ &= \frac{\alpha\sin(\pi\alpha)}{(1-\alpha)\pi} \int_0^1 h(x) \frac{x^{\alpha-1}(1-x)^{\alpha-1}}{x^{2\alpha} - 2x^\alpha(1-x)^\alpha\cos(\pi\alpha) + (1-x)^{2\alpha}}\, dx \\ &= E\big[h(\mathbb{G}_\alpha^{(3)})\big] \qquad \text{from (3.15)} \end{aligned}$$

4) We now prove (3.27)
It is shown in [BFRY], p. 319, (1.27) that :

$$\mathbb{G}_\alpha^{(3)} \overset{\text{(law)}}{=} 1 - \mathbb{G}_\alpha^{(3)} \tag{3.33}$$

which is, indeed, obvious ! Thus, from (3.26) :

$$\mathbb{G}_\alpha^{(2)} \overset{\text{(law)}}{=} \frac{\mathbb{G}_\alpha^{(3)}}{1-\mathbb{G}_\alpha^{(3)}} \overset{\text{(law)}}{=} \frac{1-\mathbb{G}_\alpha^{(3)}}{\mathbb{G}_\alpha^{(3)}} = \frac{\frac{1+\mathbb{G}_\alpha^{(2)}-\mathbb{G}_\alpha^{(2)}}{1+\mathbb{G}_\alpha^{(2)}}}{\frac{\mathbb{G}_\alpha^{(2)}}{1+\mathbb{G}_\alpha^{(2)}}} \overset{\text{(law)}}{=} \frac{1}{\mathbb{G}_\alpha^{(2)}}$$

5) The relation (3.28) follows immediately from (3.26) and from (3.17) and (3.18).

6) <u>We prove (3.29)</u>
It is shown in [BFRY], p. 320, that :

$$\mathbb{G}_\alpha^{(3)} \overset{\text{(law)}}{=} \frac{(T_{1-\alpha})^{\frac{1-\alpha}{\alpha}}}{(T'_{1-\alpha})^{\frac{1-\alpha}{\alpha}} + (T_{1-\alpha})^{\frac{1-\alpha}{\alpha}}} \quad \text{and} \quad \mathbb{G}_\alpha^{(3)} \overset{\text{(law)}}{=} \frac{(M_{1-\alpha})^{\frac{1}{\alpha}}}{(M_{1-\alpha})^{\frac{1}{\alpha}} + (M'_{1-\alpha})^{\frac{1}{\alpha}}} \tag{3.34}$$

Thus, from (3.26) and (3.34) :

$$\mathbb{G}_\alpha^{(2)} \overset{\text{(law)}}{=} \frac{\mathbb{G}_\alpha^{(3)}}{1-\mathbb{G}_\alpha^{(3)}} = \frac{\frac{(T_{1-\alpha})^{\frac{1-\alpha}{x}}}{(T'_{1-\alpha})^{\frac{1-\alpha}{\alpha}}+(T_{1-\alpha})^{\frac{1-\alpha}{\alpha}}}}{\frac{(T'_{1-\alpha})^{\frac{1-\alpha}{\alpha}}}{(T'_{1-\alpha})^{\frac{1-\alpha}{\alpha}}+(T_{1-\alpha})^{\frac{1-\alpha}{\alpha}}}} = \left(\frac{T_{1-\alpha}}{T'_{1-\alpha}}\right)^{\frac{1-\alpha}{\alpha}} \tag{3.35}$$

We note that (3.35) implies (3.27) and that (3.30) may be obtained from (3.35), in the same manner as (3.35).

7) <u>We now prove point 2 of Theorem 6</u>
The formula (3.21) $D_{1-\alpha}^{(2)} \overset{\text{(law)}}{=} \frac{\mathfrak{e}}{\gamma_\alpha}$ is an immediate consequence of (3.19) :

$$Y^{(2)} \overset{\text{(law)}}{=} \mathfrak{e} \cdot \frac{\gamma_{1-\alpha}}{\gamma_\alpha} \overset{\text{(law)}}{=} \gamma_{1-\alpha} D_{1-\alpha}^{(2)}$$

after observing that, in the latter formula, we may "simplify by $\gamma_{1-\alpha}$ $\big($see [C.Y] or [JRY], point 1.4.6 for a justification of this "simplification"$\big)$. The value of the density of $D_{1-\alpha}^{(2)}$ which is given by (3.24) now follows easily from $D_{1-\alpha}^{(2)} \overset{\text{(law)}}{=} \frac{\mathfrak{e}}{\gamma_\alpha}$. Finally, we have :

$$\begin{aligned} E(e^{-\lambda D_{1-\alpha}^{(2)}}) &= E(e^{-\lambda \frac{\mathfrak{e}}{\gamma_\alpha}}) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \int_0^\infty e^{-\lambda \frac{x}{y} - x - y} y^{\alpha-1} dx\, dy \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-y} y^\alpha dy \int_0^\infty e^{-z(\lambda+y)} dz \left(\text{after making the change of variable } \frac{x}{y} = z\right) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{y^\alpha}{(\lambda+y)} e^{-y} dy \end{aligned} \tag{3.36}$$

The formula :

$$E(e^{-\lambda D_{1-\alpha}^{(2)}}) = \alpha \int_0^\infty e^{-\lambda y} \frac{dy}{(1+y)^{\alpha+1}} \tag{3.37}$$

follows immediately from (3.24) and it is easy to verify that :

$$\frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-y} \frac{y^\alpha}{\lambda+y}\, dy = \alpha \int_0^\infty e^{-\lambda y} \frac{dy}{(1+y)^{\alpha+1}}$$

Indeed :

$$\begin{aligned} \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-y} \frac{y^\alpha}{\lambda+y}\, dy &= \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-y} y^\alpha dy \int_0^\infty e^{-z(\lambda+y)} dz \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-\lambda z} dz \int_0^\infty e^{-y(1+z)} y^\alpha dy = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \int_0^\infty e^{-\lambda z} \frac{dz}{(1+z)^{\alpha+1}} \\ &= \alpha \int_0^\infty e^{-\lambda z} \frac{dz}{(1+z)^{\alpha+1}} \end{aligned}$$

This ends the proof of Theorem 6.

Remark 7

1) From the relation $Y^{(2)} \overset{(\text{law})}{=} \gamma_{1-\alpha} D^{(2)}_{1-\alpha}$, we deduce :

$$\begin{aligned}
E(e^{-\lambda Y^{(2)}}) &= E(e^{-\lambda \gamma_{1-\alpha} \cdot D^{(2)}_{1-\alpha}}) = E\left(\frac{1}{\left(1+\lambda D^{(2)}_{1-\alpha}\right)^{1-\alpha}}\right) \\
&= \alpha \int_0^\infty \left(\frac{1+\lambda x}{1+x}\right)^{\alpha-1} \frac{dx}{(1+x)^2} \qquad \big(\text{from (3.24)}\big) \\
&= \frac{\alpha}{\lambda-1} \int_1^\lambda y^{\alpha-1} dy \quad \left(\text{after making the change of variable } \frac{1+\lambda x}{1+x} = y\right) \\
&= \frac{\lambda^\alpha - 1}{\lambda - 1}
\end{aligned}$$

This is another way to obtain (3.20).

2) Here is now another way to obtain (3.23). It is clear, from (3.25) that $E\big(|\log \mathbb{G}^{(2)}_\alpha|\big) < \infty$ and, since $\mathbb{G}^{(2)}_\alpha \overset{(\text{law})}{=} \dfrac{1}{\mathbb{G}^{(2)}_\alpha}$, that $E(\log \mathbb{G}^{(2)}_\alpha) = 0$. Thus, from Theorem 2.1, point *ii)* in [JRY], we have :

$$f_{Y^{(2)}}(u) = \frac{u^{-\alpha}}{\Gamma(1-\alpha)}\, E(e^{-u\, D^{(2)}_{1-\alpha}})$$

$\left(\text{this is formula (2.7) in [JRY], with } t = 1-\alpha,\ E(\log G) = 0 \text{ and } G \overset{(\text{law})}{=} \dfrac{1}{G}\right)$. Hence, since :

$$f_{Y^{(2)}}(u) = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty e^{-ux} \frac{x^\alpha}{1+x}\, dx = \frac{u^{-\alpha}\sin(\pi\alpha)}{\pi} \int_0^\infty e^{-y} \frac{y^\alpha}{u+y}\, dy$$

(after the change of variable $ux = y$), we obtain :

$$\begin{aligned}
E(e^{-u\, D^{(2)}_{1-\alpha}}) &= \frac{\sin(\pi\alpha)}{\pi}\, \Gamma(1-\alpha) \int_0^\infty e^{-y} \frac{y^\alpha}{u+y}\, dy \\
&= \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-y} \frac{y^\alpha}{u+y}\, dy
\end{aligned}$$

3) Furthermore, we remark that, from Theorem 2.1 of [JRY] :

$$f_{D^{(2)}_{1-\alpha}}(u) = u^{-\alpha-1} f_{D^{(2)}_{1-\alpha}}\left(\frac{1}{u}\right)$$

This formula follows also from (3.24).

4) Finally, we also observe, from Theorem 2.1 in [JRY], as a consequence of $\mathbb{G}^{(2)}_\alpha \overset{(\text{law})}{=} \dfrac{1}{\mathbb{G}^{(2)}_\alpha}$ and $E(\log \mathbb{G}^{(2)}_\alpha) = 0$, that :

$$f_{Y^{(2)}}(u) = E\left[\left(\frac{Y^{(2)}}{u}\right)^{\frac{\alpha}{2}} J_{-\alpha}\left(2\sqrt{u\, Y^{(2)}}\right)\right] \tag{3.38}$$

where $J_{-\alpha}$ denotes the Bessel function with index $(-\alpha)$.

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